

MORE ON TIE-POINTS AND HOMEOMORPHISM IN \mathbb{N}^*

ALAN DOW AND SAHARON SHELAH

ABSTRACT. A point x is a (bow) tie-point of a space X if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with x in its closure. We picture (and denote) this as $X = A \bowtie_x B$ where A, B are the closed sets which have a unique common accumulation point x . Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$ (e.g. [10, 7]) and in the recent study [4, 2] of (precisely) 2-to-1 maps on \mathbb{N}^* . In these cases the tie-points have been the unique fixed point of an involution on \mathbb{N}^* . One application of the results in this paper is the consistency of there being a 2-to-1 continuous image of \mathbb{N}^* which is not a homeomorph of \mathbb{N}^* .

1. INTRODUCTION

A point x is a tie-point of a space X if there are closed sets A, B of X such that $\{x\} = A \cap B$ and x is an adherent point of both A and B . We let $X = A \bowtie_x B$ denote this relation and say that x is a tie-point as witnessed by A, B . Let $A \equiv_x B$ mean that there is a homeomorphism from A to B with x as a fixed point. If $X = A \bowtie_x B$ and $A \equiv_x B$, then there is an involution F of X (i.e. $F^2 = \text{id}$) such that $\{x\} = \text{fix}(F)$. In this case we will say that x is a symmetric tie-point of X .

An autohomeomorphism F of \mathbb{N}^* is said to be *trivial* if there is a bijection f between cofinite subsets of \mathbb{N} such that $F = \beta f \upharpoonright \mathbb{N}^*$. Since the fixed point set of a trivial autohomeomorphism is clopen, a symmetric tie-point gives rise to a non-trivial autohomeomorphism.

If A and B are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \bowtie_{x=y} B$ denote the quotient space of

Date: February 2, 2008.

1991 Mathematics Subject Classification. 03A35.

Key words and phrases. automorphism, Stone-Cech, fixed points.

Research of the first author was supported by NSF grant No. NSF-. The research of the second author was supported by The Israel Science Foundation founded by the Israel Academy of Sciences and Humanities, and by NSF grant No. NSF-. This is paper number 917 in the second author's personal listing.

$A \oplus B$ obtained by identifying x and y and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

In this paper we establish the following theorem.

Theorem 1.1. *It is consistent that \mathbb{N}^* has symmetric tie-points x, y as witnessed by A, B and A', B' respectively such that \mathbb{N}^* is not homeomorphic to the space $A \underset{x=y}{\boxtimes} A'$*

Corollary 1.1. *It is consistent that there is a 2-to-1 image of \mathbb{N}^* which is not a homeomorph of \mathbb{N}^* .*

One can generalize the notion of tie-point and, for a point $x \in \mathbb{N}^*$, consider how many disjoint clopen subsets of $\mathbb{N}^* \setminus \{x\}$ (each accumulating to x) can be found. Let us say that a tie-point x of \mathbb{N}^* satisfies $\tau(x) \geq n$ if $\mathbb{N}^* \setminus \{x\}$ can be partitioned into n many disjoint clopen subsets each accumulating to x . Naturally, we will let $\tau(x) = n$ denote that $\tau(x) \geq n$ and $\tau(x) \not\geq n+1$. Each point x of character ω_1 in \mathbb{N}^* is a symmetric tie-point and satisfies that $\tau(x) \geq n$ for all n . We list several open questions in the final section.

More generally one could study the symmetry group of a point $x \in \mathbb{N}^*$: e.g. set G_x to be the set of autohomeomorphisms F of \mathbb{N}^* that satisfy $\text{fix}(F) = \{x\}$ and two are identified if they are the same on some clopen neighborhood of x .

Theorem 1.2. *It is consistent that \mathbb{N}^* has a tie-point x such that $\tau(x) = 2$ and such that with $\mathbb{N}^* = A \underset{x}{\boxtimes} B$, neither A nor B is a homeomorph of \mathbb{N}^* . In addition, there are no symmetric tie-points.*

The following partial order \mathbb{P}_2 , was introduced by Velickovic in [10] to add a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ while doing as little else as possible — at least assuming PFA.

Definition 1.1. The partial order \mathbb{P}_2 is defined to consist of all 1-to-1 functions $f : A \rightarrow B$ where

- $A \subseteq \omega$ and $B \subseteq \omega$
- for all $i \in \omega$ and $n \in \omega$, $f(i) \in (2^{n+1} \setminus 2^n)$ if and only if $i \in (2^{n+1} \setminus 2^n)$
- $\limsup_{n \rightarrow \omega} |(2^{n+1} \setminus 2^n) \setminus A| = \omega$ and hence, by the previous condition, $\limsup_{n \rightarrow \omega} |(2^{n+1} \setminus 2^n) \setminus B| = \omega$

The ordering on \mathbb{P}_2 is \subseteq^* .

We define some trivial generalizations of \mathbb{P}_2 . We use the notation \mathbb{P}_2 to signify that this poset introduces an involution of \mathbb{N}^* because the conditions $g = f \cup f^{-1}$ satisfy that $g^2 = g$. In the definition of \mathbb{P}_2 it is possible to suppress mention of A, B (which we do) and

to have the poset \mathbb{P}_2 consist simply of the functions g (and to treat $A = \min(g) = \{i \in \text{dom}(g) : i < g(i)\}$ and to treat B as $\max(g) = \{i \in \text{dom}(g) : g(i) < i\}$).

Let \mathbb{P}_1 denote the poset we get if we omit mention of f but consisting only of disjoint pairs (A, B) , satisfying the growth condition in Definition 1.1, and extension is coordinatewise mod finite containment. To be consistent with the other two posets, we may instead represent the elements of \mathbb{P}_1 as partial functions into 2.

More generally, let \mathbb{P}_ℓ be similar to \mathbb{P}_2 except that we assume that conditions consist of functions g satisfying that $\{i, g(i), g^2(i), \dots, g^\ell(i)\}$ has precisely ℓ elements for all $i \in \text{dom}(g)$ (and replace the intervals $2^{n+1} \setminus 2^n$ by $\ell^{n+1} \setminus \ell^n$ in the definition).

The basic properties of \mathbb{P}_2 as defined by Velickovic and treated by Shelah and Steprans are also true of \mathbb{P}_ℓ for all $\ell \in \mathbb{N}$.

In particular, for example, it is easily seen that

Proposition 1.1. *If $L \subset \mathbb{N}$ and $\mathbb{P}^* = \prod_{\ell \in L} \mathbb{P}_\ell$ (with full supports) and G is a \mathbb{P}^* -generic filter, then in $V[G]$, for each $\ell \in L$, there is a tie-point $x_\ell \in \mathbb{N}^*$ with $\tau(x_\ell) \geq \ell$.*

For the proof of Theorem 1.1 we use $\mathbb{P}_2 \times \mathbb{P}_2$ and for the proof of Theorem 1.2 we use \mathbb{P}_1 .

An ideal \mathcal{I} on \mathbb{N} is said to be ccc over fin [3], if for each uncountable almost disjoint family, all but countably many of them are in \mathcal{I} . An ideal is a P -ideal if it is countably directed closed mod finite.

The following main result is extracted from [6] and [8] which we record without proof.

Lemma 1.1 (PFA). *If \mathbb{P}^* is a finite or countable product (repetitions allowed) of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$ and if G is a \mathbb{P}^* -generic filter, then in $V[G]$ every autohomeomorphism of \mathbb{N}^* has the property that the ideal of sets on which it is trivial is a P -ideal which is ccc over fin .*

Corollary 1.2 (PFA). *If \mathbb{P}^* is a finite or countable product of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$, and if G is a \mathbb{P}^* -generic filter, then in $V[G]$ if F is an autohomeomorphism of \mathbb{N}^* and $\{Z_\alpha : \alpha \in \omega_2\}$ is an increasing mod finite chain of infinite subsets of \mathbb{N} , there is an $\alpha_0 \in \omega_2$ and a collection $\{h_\alpha : \alpha \in \omega_2\}$ of 1-to-1 functions such that $\text{dom}(h_\alpha) = Z_\alpha$ and for all $\beta \in \omega_2$ and $a \subset Z_\beta \setminus Z_{\alpha_0}$, $F[a] =^* h_\beta[a]$.*

Each poset \mathbb{P}^* as above is \aleph_1 -closed and \aleph_2 -distributive (see [8, p.4226]). In this paper we will restrict our study to finite products. The following partial order can be used to show that these products are \aleph_2 -distributive.

Definition 1.2. Let \mathbb{P}^* be a finite product of posets from $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$. Given $\{\vec{f}_\xi : \xi \in \mu\} = \mathfrak{F} \subset \mathbb{P}^*$ (decreasing in the ordering on \mathbb{P}^*), define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $g \in \mathbb{P}^*$ such that there is some $\xi \in \mu$ such that $\vec{g} \equiv^* \vec{f}_\xi$. The ordering on $\mathbb{P}(\mathfrak{F})$ is coordinatewise \supseteq as opposed to $^*\supseteq$ in \mathbb{P}^* .

Corollary 1.3 (PFA). *If \mathbb{P}^* is a finite or countable product of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$, and if G is a \mathbb{P}^* -generic filter, then in $V[G]$ if F is an involution of \mathbb{N}^* with a unique fixed point x , then x is a P_{ω_2} -point and $\mathbb{N}^* = A \underset{x}{\bowtie} B$ for some A, B such that $F[A] = B$.*

Proof. We may assume that F also denotes an arbitrary lifting of F to $[\mathbb{N}]^\omega$ in the sense that for each $Y \subset \mathbb{N}$, $(F[Y])^* = F[Y^*]$. Let $\mathcal{Z}_x = [\mathbb{N}]^\omega \setminus x$ (the dual ideal to x). For each $Z \in \mathcal{Z}_x$, $F[Z]$ is also in \mathcal{Z} and $F[Z \cup F[Z]] =^* Z \cup F[Z]$. So let us now assume that \mathcal{Z} denotes those $Z \in \mathcal{Z}_x$ such that $Z =^* F[Z]$. Given $Z \in \mathcal{Z}$, since $\text{fix}(F) \cap Z^* = \emptyset$, there is a collection $\mathcal{Y} \subset [Z]^\omega$ such that $F[Y] \cap Y =^* \emptyset$ for each $Y \in \mathcal{Y}$, and such that Z^* is covered by $\{Y^* : Y \in \mathcal{Y}\}$. By compactness, we may assume that $\mathcal{Y} = \{Y_0, \dots, Y_n\}$ is finite. Set $Z_0 = Y_0 \cup F[Y_0]$. By induction, replace Y_k by $Y_k \setminus \bigcup_{j < k} Z_j$ and define $Z_k = Y_k \cup F[Y_k]$. Therefore $Y_Z = \bigcup_k Y_k$ satisfies that $Y_Z \cap F[Y_Z] =^* \emptyset$ and $Z = Y_Z \cup F[Y_Z]$. This shows that for each $Z \in \mathcal{Z}$ there is a partition of $Z = Z^0 \cup Z^1$ such that $F[Z^0] =^* Z^1$. It now follows that x is a P -point, for if $\{Z_n = Z_n^0 \cup Z_n^1 : n \in \mathbb{N}\} \subset \mathcal{Z}$ are pairwise disjoint, then $x \notin \overline{\bigcup_n Z_n^*}$ since $F[\overline{\bigcup_n (Z_n^0)^*}] = \overline{\bigcup_n (Z_n^1)^*}$ and $\overline{\bigcup_n (Z_n^0)^*}$ is disjoint from $\overline{\bigcup_n (Z_n^1)^*}$.

Now we prove that it is a P_{ω_2} -point. Assume that $\{Z_\alpha : \alpha \in \omega_1\} \subset \mathcal{Z}$ is a mod finite increasing sequence. By Lemma 1.1 (similar to Corollary 1.2) we may assume, by possibly removing some Z_{α_0} from each Z_α , that there is a sequence $\{h_\alpha : \alpha \in \omega_1\}$ of involutions such that h_α induces $F \upharpoonright Z_\alpha^*$. For each $\alpha \in \omega_1$, let $a_\alpha = \min(h_\alpha) = \{i \in \text{dom}(h_\alpha) : i < h_\alpha(i)\}$ and $b_\alpha = Z_\alpha \setminus a_\alpha$. It follows that $F[a_\alpha] =^* b_\alpha$. Since \mathbb{P}^* is \aleph_2 -distributive, all of these \aleph_1 -sized sets are in V which is a model of PFA. If x is in the closure of $\bigcup_{\alpha \in \omega_1} Z_\alpha^*$, then x is in the closure of each of $\bigcup_\alpha a_\alpha^*$ and $\bigcup_\alpha b_\alpha^*$. Therefore, it suffices to show that $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ can not form a gap in V . As is well-known, if \mathcal{A} does form a gap, there is a ccc poset $Q_{\mathcal{A}}$ which adds an uncountable I such that $\{(a_\alpha, b_\alpha) : \alpha \in I\}$ forms a Hausdorff-gap (i.e. *freezes* the gap). It is easy to prove that if \mathbb{C} is the poset for adding ω_1 -many almost disjoint Cohen reals, $\{\dot{C}_\xi : \xi \in \omega_1\}$, then a similar ccc poset $\mathbb{C} * \dot{Q}$ will introduce, for each $\xi \in \omega_1$, an uncountable $I_\xi \subset \omega_1$, such that $\{(\dot{C}_\xi \cap a_\alpha, \dot{C}_\xi \cap b_\alpha) : \alpha \in I_\xi\}$ is a Hausdorff-gap. But now by Lemma

1.1, it follows that there is some $\xi \in \omega_1$ such that $Z = C_\xi \in \mathcal{Z}$, $F \restriction Z^*$ is trivial and for some uncountable $I \subset \omega_1$ $\{(Z \cap a_\alpha, Z \cap b_\alpha) : \alpha \in I\}$ forms a Hausdorff-gap. This however is a contradiction because if h_Z induces $F \restriction Z^*$, then $\min(h_Z) \cap ((Z \cap a_\alpha) \cup (Z \cap b_\alpha))$ is almost equal to a_α for all $\alpha \in \omega_1$, i.e. $\min(h_Z)$ would have to split the Hausdorff-gap. \square

The forcing $\mathbb{P}(\mathfrak{F})$ introduces a tuple \vec{f} which satisfies $\vec{f} \leq \vec{f}_\alpha$ for $\vec{f}_\alpha \in \mathfrak{F}$ but for the fact that \vec{f} may not be a member of \mathbb{P}^* simply because the domains of the component functions are too big. There is a σ -centered poset which will choose an appropriate sequence \vec{f}^* of subfunctions of \vec{f} which is a member of \mathbb{P}^* and which is still below each member of \mathfrak{F} (see [6, 2.1]).

A strategic choice of the sequence \mathfrak{F} will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc, but remarkably even more is true. Again we are lifting results from [6, 2.6] and [8, proof of Thm. 3.1]. This is an innovative factoring of Velickovic's original amoeba forcing poset and seems to preserve more properties. Let $\omega_2^{<\omega_1}$ denote the standard collapse which introduces a function from ω_1 onto ω_2 .

Lemma 1.2. *Let \mathbb{P}^* be a finite product of posets from $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$. In the forcing extension, $V[H]$, by $\omega_2^{<\omega_1}$, there is a descending sequence \mathfrak{F} from \mathbb{P}^* which is \mathbb{P}^* -generic over V and, for which, $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω -bounding.*

It follows also that $\mathbb{P}(\mathfrak{F})$ preserves that $\mathbb{R} \cap V$ is of second category. This was crucial in the proof of Lemma 1.1. We can manage with the ω^ω -bounding property because we are going to use Lemma 1.1. A poset is said to be ω^ω -bounding if every new function in ω^ω is bounded by some ground model function.

The following proposition is probably well-known but we do not have a reference.

Proposition 1.2. *Assume that \mathbb{Q} is a ccc ω^ω -bounding poset and that x is an ultrafilter on \mathbb{N} . If G is a \mathbb{Q} -generic filter then there is no set $A \subset \mathbb{N}$ such that $A \setminus Y$ is finite for all $Y \in x$.*

Proof. Assume that $\{\dot{a}_n : n \in \omega\}$ are \mathbb{Q} -names of integers such that $1 \Vdash_{\mathbb{Q}} \dot{a}_n \geq n$. Let A denote the \mathbb{Q} -name so that $\Vdash_{\mathbb{Q}} A = \{\dot{a}_n : n \in \omega\}$. Since \mathbb{Q} is ω^ω -bounding, there is some $q \in \mathbb{Q}$ and a sequence $\{n_k : k \in \omega\}$ in V such that $q \Vdash_{\mathbb{Q}} "n_k \leq \dot{a}_i \leq n_{k+2} \ \forall i \in [n_k, n_{k+1})"$. There is some $\ell \in 3$ such that $Y = \bigcup_k [n_{3k+\ell}, n_{3k+\ell+1})$ is a member of x . On the other hand, $q \Vdash_{\mathbb{Q}} "A \cap [n_{3k+\ell+1}, n_{3k+\ell+3})"$ is not empty for each k . Therefore $q \not\Vdash_{\mathbb{Q}} "A \setminus Y \text{ is finite}"$. \square

Another interesting and useful general lemma is the following.

Lemma 1.3. *Let $\mathcal{F} \subset \mathbb{P}_\ell$ (for any $\ell \in \mathbb{N}$) be generic over V , then for each $\mathbb{P}(\mathfrak{F})$ -name $\dot{h} \in \mathbb{N}^\mathbb{N}$, either there is an $f \in \mathfrak{F}$ such that $f \Vdash_{\mathbb{P}(\mathfrak{F})} \dot{h} \restriction \text{dom}(f) \notin V$, or there is an $f \in \mathfrak{F}$ and an increasing sequence $n_0 < n_1 < \dots$ of integers such that for each $i \in [n_k, n_{k+1})$ and each $g < f$ such that g forces a value on $\dot{h}(i)$, $f \cup (g \restriction [n_k, n_{k+1}))$ also forces a value on $\dot{h}(i)$.*

Proof. Given any f , perform a standard fusion (see [6, 2.4] or [8, 3.4]) f_k, n_k by picking $L_k \subset [n_{k+1}, n_{k+2})$ (absorbed into $\text{dom}(f_{k+1})$) so that for each partial function s on n_k which extends $f_k \restriction n_k$, if there is some integer $i \geq n_{k+1}$ for which no $< n_k$ -preserving extension of $s \cup f_k$ forces a value on $\dot{h}(i)$, then there is such an integer in $\text{dom}(f_{k+1})$. Let \bar{f} be the fusion and note that either \bar{f} forces that $\dot{h} \restriction \text{dom}(\bar{f})$ is not in V , or it forces that our sequence of n_k 's does the job. Thus, we have proven that for each f , there is such a \bar{f} , hence by genericity, there is such an \bar{f} in \mathfrak{F} . \square

2. PROOF OF THEOREM 1.1

Theorem 2.1 (PFA). *If G is a generic filter for $\mathbb{P}^* = \mathbb{P}_2 \times \mathbb{P}_2$, then there are symmetric tie-points x, y as witnessed by A, B and C, D respectively such that \mathbb{N}^* is not homeomorphic to the space $A \underset{x=y}{\times} C$*

Assume that \mathbb{N}^* is homeomorphic to $A \underset{x=y}{\times} C$ and that z is the \mathbb{P}^* -name of the ultrafilter that is sent (by the assumed homeomorphism) to the point (x, y) in the quotient space $A \underset{x=y}{\times} C$.

Further notation: let $\{a_\alpha : \alpha \in \omega_2\}$ be the \mathbb{P}_2 -names of the infinite subsets of \mathbb{N} which form the mod-finite increasing chain whose remainders in \mathbb{N}^* cover $A \setminus \{x\}$ and, similarly let $\{c_\alpha : \alpha \in \omega_2\}$ be the \mathbb{P}_2 -names (second coordinates though) which form the chain in $C \setminus \{y\}$.

If we represent $A \underset{x=y}{\times} C$ as a quotient of $(\mathbb{N} \times 2)^*$, we may assume that F is a \mathbb{P}^* -name of a function from $[\mathbb{N}]^\omega$ into $[\mathbb{N} \times 2]^\omega$ such that letting $Z_\alpha = F^{-1}(a_\alpha \times \{0\} \cup c_\alpha \times \{1\})$ for each $\alpha \in \omega_2$, then $\{Z_\alpha : \alpha \in \omega_2\}$ forms the dual ideal to z , and $F : [Z_\alpha]^\omega \rightarrow (a_\alpha \times \{0\} \cup c_\alpha \times \{1\})^\omega$ induces the above homeomorphism from Z_α^* onto $(a_\alpha^* \times \{0\}) \cup (c_\alpha^* \times \{1\})$.

By Corollary 1.2, we may assume that for each $\beta \in \omega_2$, there is a bijection h_β between some cofinite subset of Z_β and some cofinite subset of $(a_\beta \times \{0\}) \cup (c_\beta \times \{1\})$ which induces $F \restriction [Z_\beta]^\omega$ (since we can just ignore Z_{α_0} for some fixed α_0). We will use $F \restriction [Z_\beta]^\omega = h_\beta$ to mean that h_β induces $F \restriction [Z_\beta]^\omega$. Note that by the assumptions, for each $\beta \in \omega_2$, there is a $\gamma \in \omega_2$ such that each of $h_\gamma^{-1}(a_\gamma) \setminus Z_\beta$ and $h_\gamma^{-1}(c_\gamma) \setminus Z_\beta$ are infinite.

Let H be a generic filter for $\omega_2^{<\omega_1}$, and assume that $\mathfrak{F} \subset \mathbb{P}^*$ is chosen as in Lemma 1.2. In this model, let us use λ to denote the ω_2 from V . Using the fact that \mathfrak{F} is \mathbb{P}^* -generic over V , we may treat all the functions h_α ($\alpha \in \lambda$) as members of V since we can take the valuation of all the \mathbb{P}^* -names using \mathfrak{F} . Assume that \dot{h} is a $\mathbb{P}(\mathfrak{F})$ -name of a finite-to-1 function from \mathbb{N} into $\mathbb{N} \times 2$ satisfying that $h_\alpha \subset^* \dot{h}$ for all $\alpha \in \lambda$. We show there is no such \dot{h} .

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω -bounding, there is an increasing sequence of integers $\{n_k : k \in \omega\}$ and an $\vec{f}_0 = (g_0, g_1) \in \mathfrak{F}$ such that

- (1) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}\dot{h}(i) \in ([0, n_{k+2}) \times 2)\text{"}$
- (2) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}\dot{h}^{-1}(\{i\} \times 2) \subset [0, n_{k+2})\text{"}$
- (3) for each k and each $j \in \{0, 1\}$ there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \text{dom } g_j$ has at least k elements.

Choose any $(g'_0, g'_1) = \vec{f}_1 < \vec{f}_0$ such that $\mathbb{N} \setminus \text{dom}(g'_0) \subset \bigcup_k [n_{6k+1}, n_{6k+2})$ and $\mathbb{N} \setminus \text{dom}(g'_1) \subset \bigcup_k [n_{6k+4}, n_{6k+5})$. Next, choose any $\vec{f}_2 < \vec{f}_1$ and some $\alpha \in \lambda$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}\text{dom}(g'_0) \subset^* a_\alpha \cup g'_0[a_\alpha] \text{ and } \text{dom}(g'_1) \subset^* c_\alpha \cup g'_1[c_\alpha]\text{"}$. For each $\gamma \in \lambda$, note that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}a_\gamma \setminus a_\alpha \subset^* \mathbb{N} \setminus \text{dom}(g'_0)\text{"}$ and similarly $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}c_\gamma \setminus c_\alpha \subset^* \mathbb{N} \setminus \text{dom}(g'_1)\text{"}$.

Now consider the two disjoint sets: $Y_0 = \bigcup_k [n_{6k}, n_{6k+3})$ and $Y_1 = \bigcup_k [n_{6k+3}, n_{6k+6})$. Since z is an ultrafilter in this extension, by possibly extending \vec{f}_2 even more, we may assume that there is some $j \in \{0, 1\}$ and some $\beta > \alpha$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}Y_j \subset^* Z_\beta\text{"}$. Without loss of generality (by symmetry) we may assume that $j = 0$. Consider any $\gamma \in \lambda$. Since we are assuming that $h_\gamma \subset^* \dot{h}$, we have that \vec{f}_2 forces that $h_\gamma[Z_\gamma \setminus Z_\alpha] =^* \dot{h}[Z_\gamma \setminus Z_\alpha]$. We also have that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{"}\dot{h}[Y_0] \ast \supset (a_\gamma \setminus a_\alpha) \times \{0\} =^* h_\gamma[Z_\gamma \setminus Z_\alpha] \cap \mathbb{N} \times \{0\}\text{"}$. Putting this all together, we now have that \vec{f}_2 forces that $\dot{h}[Z_\beta]$ almost contains $(a_\gamma \setminus a_\alpha) \times \{0\}$ for all $\gamma \in \lambda$; which clearly contradicts that $\dot{h}[Z_\beta]$ is supposed to be almost equal to $h_\beta[Z_\beta]$.

So now what? Well, let H_2 be a generic filter for $\mathbb{P}(\mathfrak{F})$ and consider the family of functions $\mathcal{H}_\lambda = \{h_\alpha : \alpha \in \lambda\}$ which we know does not have a common finite-to-1 extension.

Before proceeding, we need to show that \mathcal{H}_λ does not have any extension h . If \dot{h} is any $\mathbb{P}(\mathfrak{F})$ -name of a function for which it is forced that $h_\alpha \subset^* \dot{h}$ for all $\alpha \in \lambda$, then there is some $\ell \in \mathbb{N}$ such that $\dot{Y} = h^{-1}(\dot{h}(\ell))$ is (forced to be) infinite. It follows easily that \dot{Y} is forced to be almost contained in every member of z . By Lemma 1.2 this cannot happen. Therefore the family \mathcal{H}_λ does not have any common extension.

Given such a family as \mathcal{H}_λ , there is a well-known proper poset \dot{Q}_1 (see [1, 3.1], [3, 2.2.1], and [10, p9]) which will force an uncountable cofinal $I \subset \lambda$ and a collection of integers $\{k_{\alpha,\beta} : \alpha < \beta \in I\}$ satisfying that $h_\alpha(k_{\alpha,\beta}) \neq h_\beta(k_{\alpha,\beta})$ (and both are defined) for $\alpha < \beta \in I$. So, let \dot{Q}_1 be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F})$ -name of the above mentioned poset. In addition, let $\dot{\varphi}$ be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$ -name of the enumerating function from ω_1 onto I , and let $\dot{k}_{\alpha,\beta}$ (for $\alpha < \beta \in \omega_2$) be the name of the integer $k_{\dot{\varphi}(\alpha), \dot{\varphi}(\beta)}$. Thus for each $\alpha < \beta \in \omega_1$, there is a dense set $D(\alpha, \beta) \subset \omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$ such that for each member p of $D(\alpha, \beta)$, there are functions h_α, h_β in V and sets $Z_\alpha = \text{dom}(h_\alpha)$, $Z_\beta = \text{dom}(h_\beta)$ and integers $k = k(\alpha, \beta) \in Z_\alpha \cap Z_\beta$ such that

$$p \Vdash_{\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1} "F \restriction [Z_\alpha]^\omega = h_\alpha, F \restriction [Z_\beta]^\omega = h_\beta, h_\alpha(k) \neq h_\beta(k)".$$

Finally, let \dot{Q}_2 be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$ -name of the σ -centered poset which forces an element $\vec{f} \in \mathbb{P}^*$ which is below every member of \mathfrak{F} . Again, there is a countable collection of dense subsets of the proper poset $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1 * \dot{Q}_2$ which determine the values of \vec{f} .

Applying PFA to the above proper poset and the family of ω_1 mentioned dense sets, we find there is a sequence $\{h'_\alpha, Z'_\alpha : \alpha \in \omega_1\}$, integers $\{k_{\alpha,\beta} : \alpha < \beta \in \omega_1\}$, and a condition $\vec{f} \in \mathbb{P}^*$ such that, for all $\alpha < \beta$ and $k = k(\alpha, \beta)$,

$$\vec{f} \Vdash_{\mathbb{P}^*} "F \restriction [Z'_\alpha]^\omega = h'_\alpha, F \restriction [Z'_\beta]^\omega = h'_\beta, h'_\alpha(k) \neq h'_\beta(k)".$$

But, we also know that we can choose \vec{f} so that there is some $\lambda \in \omega_2$, and some h_λ, Z_λ such that, for all $\alpha \in \omega_1$, $Z'_\alpha \subset^* Z_\lambda$ and $F \restriction [Z_\lambda]^\omega = h_\lambda$.

It follows of course that for all $\alpha \in \omega_1$, there is some n_α such that $h'_\alpha \restriction [n_\alpha, \omega) \subset h_\lambda$. Let $J \in [\omega_1]^{\omega_1}$, $n \in \omega$, and h' a function with $\text{dom}(h') \subset n$ such that $n_\alpha = n$ and $h'_\alpha \restriction n = h'$ for all $\alpha \in J$. We now have a contradiction since if $\alpha < \beta \in J$ then clearly $k = k(\alpha, \beta) \geq n$ and this contradicts that $h'_\alpha(k)$ and $h'_\beta(k)$ are both supposed to equal $h_\lambda(k)$.

3. PROOF OF THEOREM 1.2

Theorem 3.1 (PFA). *If G is a generic filter for \mathbb{P}_1 , then a tie-point x is introduced such that $\tau(x) = 2$ and with $\mathbb{N}^* = A \mathbin{\overset{x}{\times}} B$, neither A nor B is a homeomorph of \mathbb{N}^* . In addition, there is no involution F on \mathbb{N}^* which has a unique fixed point, and so, no tie-point is symmetric.*

Assume that V is a model of PFA and that $\mathbb{P} = \mathbb{P}_1$. The elements of \mathbb{P} are partial functions f from \mathbb{N} into 2 which also satisfy that $\limsup_{n \in \mathbb{N}} |2^{n+1} \setminus (2^n \cup \text{dom}(f))| = \infty$. The ordering on \mathbb{P} is that

$f < g$ ($f, g \in \mathbb{P}$) if $g \subset^* f$. For each $f \in \mathbb{P}$, let $a_f = f^{-1}(0)$ and $b_f = f^{-1}(1)$.

Again we assume that $\{a_\alpha : \alpha \in \omega_2\}$ is the sequence of \mathbb{P} -names satisfying that $\mathbb{N}^* = A \mathbin{\dot{\times}}_x B$ and $A \setminus \{x\} = \bigcup \{a_\alpha^* : \alpha \in \omega_2\}$. Of course by this we mean that for each $f \in G$, there are $\alpha \in \omega_2$, $a \in [\mathbb{N}]^\omega$, and $f_1 \in G$ such that $a_f \subset^* a \subset a_{f_1}$ and $f_1 \Vdash_{\mathbb{P}} \check{a} = a_\alpha$.

Next we assume that, if A is homeomorphic to \mathbb{N}^* , then F is a \mathbb{P} -name of a homeomorphism from \mathbb{N}^* to A and let z denote the point in \mathbb{N}^* which F sends to x . Also, let Z_α be the \mathbb{P} -name of $F^{-1}[a_\alpha]$ and recall that $\mathbb{N}^* \setminus \{z\} = \bigcup \{Z_\alpha^* : \alpha \in \omega_2\}$. As above, we may also assume that for each $\alpha \in \omega_2$, there is a \mathbb{P} -name of a function h_α with $\text{dom}(h_\alpha) = Z_\alpha$ such that $F \restriction [Z_\alpha]^\omega$ is induced by h_α .

Furthermore if $\tau(x) > 2$, then one of $A \setminus \{x\}$ or $B \setminus \{x\}$ can be partitioned into disjoint clopen non-compact sets. We may assume that it is $A \setminus \{x\}$ which can be so partitioned. Therefore there is some sequence $\{c_\alpha : \alpha \in \omega_2\}$ of \mathbb{P} -names such that for each $\alpha < \beta \in \omega_2$, $c_\beta \subset a_\beta$ and $c_\beta \cap a_\alpha =^* c_\alpha$. In addition, for each $\alpha < \omega_2$ there must be a $\beta \in \omega_2$ such that $c_\beta \setminus a_\alpha$ and $a_\beta \setminus (c_\beta \cup a_\alpha)$ are both infinite.

Now assume that H is $\omega_2^{<\omega_1}$ -generic and again choose a sequence $\mathfrak{F} \subset \mathbb{P}$ which is V -generic for \mathbb{P} and which forces that $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω -bounding. For the rest of the proof we work in the model $V[H]$ and we again let λ denote the ordinal ω_2^V .

In the case of \mathbb{P}_1 we are able to prove a significant strengthening of Lemma 1.3.

Lemma 3.1. *Assume that \dot{h} is a $\mathbb{P}(\mathfrak{F})$ -name of a function from \mathbb{N} to \mathbb{N} . Either there is an $f \in \mathfrak{F}$ and such that $f \Vdash_{\mathbb{P}(\mathfrak{F})} \check{h} \restriction \text{dom}(f) \notin V$, or there is an $f \in \mathfrak{F}$ and an increasing sequence $m_1 < m_2 < \dots$ of integers such that $\mathbb{N} \setminus \text{dom}(f) = \bigcup_k S_k$ where $S_k \subset 2^{m_{k+1}} \setminus 2^{m_k}$ and for each $i \in S_k$ the condition $f \cup \{(i, 0)\}$ forces a value on $\dot{h}(i)$.*

Proof. First we choose $f_0 \in \mathfrak{F}$ and some increasing sequence $n_0 < n_1 < \dots < n_k < \dots$ as in Lemma 1.3. We may choose, for each k , an m_k such that $n_k \leq 2^{m_k} < 2^{m_{k+1}} \leq n_{k+1}$ such that $\limsup_k |2^{m_{k+1}} \setminus (2^{m_k} \cup \text{dom}(f_0))| = \infty$. For each k , let $S_k^0 = 2^{m_{k+1}} \setminus (2^{m_k} \cup \text{dom}(f_0))$. By re-indexing we may assume that $|S_k^0| \geq k$, and we may arrange that $\mathbb{N} \setminus \text{dom}(f_0)$ is equal to $\bigcup_k S_k^0$ and set $L_0 = \mathbb{N}$. For each $k \in L_0$, let $i_k^0 = \min S_k^0$ and choose any $f_1' < f_0$ such that (by definition of \mathbb{P}) $I_0 = \{i_k^0 : k \in L_0\} \subset (f_1')^{-1}(0)$ and (by assumption on \dot{h}) f_1' forces a value on $\dot{h}(i_k^0)$ for each $k \in L_0$. Set $f_1 = f_1' \restriction (\mathbb{N} \setminus I_0)$ and for each $k \in L_0$, let $S_k^1 = S_k^0 \setminus (\{i_k^0\} \cup \text{dom}(f_1))$. By further extending f_1 we may also assume that $f_1 \cup \{(i_k^0, 1)\}$ also forces a value on $\dot{h}(i_k^0)$. Choose

$L_1 \subset L_0$ such that $\lim_{k \in L_1} |S_k^1| = \infty$. Notice that each member of i_k^0 is the minimum element of S_k^1 . Again, we may extend f_1 and assume that $\mathbb{N} \setminus \text{dom}(f_1)$ is equal to $\bigcup_{k \in L_1} S_k^1$. Suppose now we have some infinite L_j , some f_j , and for $k \in L_j$, an increasing sequence $\{i_k^0, i_k^1, \dots, i_k^{j-1}\} \subset S_k^0$. Assume further that

$$S_k^j \cup \{i_k^\ell : \ell < j\} = S_k^0 \setminus \text{dom}(f_j)$$

and that $\lim_{k \in L_j} |S_k^j| = \infty$. For each $k \in L_j$, let $i_k^j = \min(S_k^j \setminus \{i_k^\ell : \ell < j\})$. By a simple recursion of length 2^j , there is an $f_{j+1} < f_j$ such that, for each $k \in L_j$, $\{i_k^\ell : \ell \leq j\} \subset S_k^0 \setminus \text{dom}(f_{j+1})$ and for each function s from $\{i_k^\ell : \ell \leq j\}$ into 2, the condition $f_{j+1} \cup s$ forces a value on $\dot{h}(i_k^j)$. Again find $L_{j+1} \subset L_j$ so that $\lim_{k \in L_{j+1}} |S_k^{j+1}| = \infty$ (where $S_k^{j+1} = S_k^0 \setminus \text{dom}(f_{j+1})$) and extend f_{j+1} so that $\mathbb{N} \setminus \text{dom}(f_{j+1})$ is equal to $\bigcup_{k \in L_{j+1}} S_k^{j+1}$.

We are half-way there. At the end of this fusion, the function $\bar{f} = \bigcup_j f_j$ is a member of \mathbb{P} because for each j and $k \in L_{j+1}$, $2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(\bar{f})) \supset \{i_k^0, \dots, i_k^j\}$. For each k , let $\bar{S}_k = S_k^0 \setminus \text{dom}(\bar{f})$ and, by possibly extending \bar{f} , we may again assume that there is some L such that $\lim_{k \in L} |\bar{S}_k| = \infty$ and that, for $k \in L$, $\bar{S}_k = \{i_k^0, i_k^1, \dots, i_k^{j_k}\}$ for some j_k . What we have proven about \bar{f} is that it satisfies that for each $k \in L$ and each $j < j_k$ and each function s from $\{i_k^0, \dots, i_k^{j-1}\}$ to 2, $\bar{f} \cup s \cup (i_k^j, 0)$ forces a value on $\dot{h}(i_k^j)$.

To finish, simply repeat the process except this time choose maximal values and work down the values in \bar{S}_k . Again, by genericity of \mathfrak{F} , there must be such a condition as \bar{f} in \mathfrak{F} . \square

Returning to the proof of Theorem 3.1, we are ready to use Lemma 3.1 to show that forcing with $\mathbb{P}(\mathfrak{F})$ will not introduce undesirable functions h analogous to the argument in Theorem 1.1. Indeed, assume that we are in the case that F is a homeomorphism from \mathbb{N}^* to A as above, and that $\{h_\alpha : \alpha \in \lambda\}$ is the family of functions as above. If we show that \dot{h} does not satisfy that $h_\alpha \subset^* \dot{h}$ for each $\alpha \in \lambda$, then we proceed just as in Theorem 1.1. By Lemma 3.1, we have the condition $f_0 \in \mathfrak{F}$ and the sequence S_k ($k \in \mathbb{N}$) such that $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$ and that for each $i \in \bigcup_k S_k$, $f_0 \cup \{(i, 0)\}$ forces a value (call it $\bar{h}(i)$) on $\dot{h}(i)$. Therefore, \bar{h} is a function with domain $\bigcup_k S_k$ in V . It suffices to find a condition in \mathbb{P} below f_0 which forces that there is some α such that h_α is not extended by \dot{h} . It is useful to note that if $Y \subset \bigcup_k S_k$ is such that $\limsup |S_k \setminus Y|$ is infinite, then for any function $g \in 2^Y$, $f_0 \cup g \in \mathbb{P}$.

We first check that \bar{h} is 1-to-1 on a cofinite subset. If not, there is an infinite set of pairs $E_j \subset \bigcup_k \bar{S}_k$, $\bar{h}[E_j]$ is a singleton and such that for each k , $\bar{S}_k \cap \bigcup_j E_j$ has at most two elements. If g is the function with $\text{dom}(g) = \bigcup_j E_j$ which is constantly 0, then $f_0 \cup g$ forces that \dot{h} agrees with \bar{h} on $\text{dom}(g)$ and so is not 1-to-1. On the other hand, this contradicts that there is $f_1 < f_0 \cup g$ such that for some $\alpha \in \omega_2$, a_α almost contains $(f_0 \cup g)^{-1}(0)$ and the 1-to-1 function h_α with domain a_α is supposed to also agree with \dot{h} on $\text{dom}(g)$.

But now that we know that \bar{h} is 1-to-1 we may choose any $f_1 \in \mathfrak{F}$ such that $f_1 < f_0$ and such that there is an $\alpha \in \omega_2$ with $f_0^{-1}(0) \subset a_\alpha \subset f_1^{-1}(0)$, and f_1 has decided the function h_α . Let Y be any infinite subset of $\mathbb{N} \setminus \text{dom}(f_1)$ which meets each \bar{S}_k in at most a single point. If $\bar{h}[Y]$ meets Z_α in an infinite set, then choose $f_2 < f_1$ so that $f_2[Y] = 0$ and there is a $\beta > \alpha$ such that $Y \subset a_\beta$. In this case we will have that f_2 forces that $Y \subset a_\beta \setminus a_\alpha$, $\dot{h} \upharpoonright Y \subset^* h_\beta$, and $h_\beta[Y] \cap h_\beta[a_\alpha]$ is infinite (contradicting that h_β is 1-to-1). Therefore we must have that $\bar{h}[Y]$ is almost disjoint from Z_α . Instead consider $f_2 < f_1$ so that $f_2[Y] = 1$. By extending f_2 we may assume that there is a $\beta < \omega_2$ such that $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ “ $Z_\beta \cap \bar{h}[Y]$ is infinite”. However, since $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ “ $h_\beta \subset^* \dot{h}$ ”, we also have that $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ “ $h_\beta \upharpoonright (a_\beta \setminus a_\alpha) \subset^* \bar{h}$ and $(a_\beta \setminus a_\alpha) \cap Y =^* \emptyset$ ” contradicting that \bar{h} is 1-to-1 on $\text{dom}(f_2) \setminus \text{dom}(f_0)$. This finishes the proof that there is no $\mathbb{P}(\mathfrak{F})$ name of a function extending all the h_α ’s ($\alpha \in \lambda$) and the proof that F can not exist continues as in Theorem 1.1.

Next assume that we have a family $\{c_\alpha : \alpha \in \lambda\}$ as described above and suppose that $C = \dot{h}^{-1}(0)$ satisfies that (it is forced) $C \cap a_\alpha =^* c_\alpha$ for all $\alpha \in \lambda$. If we can show there is no such \dot{h} , then we will know that in the extension obtained by forcing with $\mathbb{P}(\mathcal{F})$, the collection $\{(c_\alpha, (a_\alpha \setminus c_\alpha)) : \alpha \in \lambda\}$ forms an (ω_1, ω_1) -gap and we can use a proper poset Q_1 to “freeze” the gap. Again, meeting ω_1 dense subsets of the iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1 * Q_2$ (where Q_2 is the σ -centered poset as in Theorem 1.1) introduces a condition $f \in \mathbb{P}$ which forces that c_λ will not exist. So, given our name \dot{h} , we repeat the steps above up to the point where we have f_0 and the sequence $\{S_k : k \in \mathbb{N}\}$ so that $f_0 \cup \{(i, 0)\}$ forces a value $\bar{h}(i)$ on $\dot{h}(i)$ for each $i \in \bigcup_k S_k$ and $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$. Let $Y = \bar{h}^{-1}(0)$ and $Z = \bar{h}^{-1}(1)$ (of course we may assume that $\bar{h}(i) \in 2$ for all i). Since x is forced to be an ultrafilter, there is an $f_1 < f_0$ such that $\text{dom}(f_1)$ contains one of Y or Z . If $\text{dom}(f_1)$ contains Y , then f_1 forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 1$ and so $(a_\beta \setminus \text{dom}(f_1)) \subset^* (\mathbb{N} \setminus C)$ for all $\beta \in \omega_2$. While if $\text{dom}(f_1)$ contains

Z , then f_1 forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 0$, and so $(a_\beta \setminus \text{dom}(f_1)) \subset^* C$ for all $\beta \in \omega_2$. However, taking β so large that each of $c_\beta \setminus \text{dom}(f_1)$ and $(a_\beta \setminus (c_\beta \cup \text{dom}(f_1)))$ are infinite shows that no such \dot{h} exists.

Finally we show that there are no involutions on \mathbb{N}^* which have a unique fixed point. Assume that F is such an involution and that z is the unique fixed point of F . Applying Corollaries 1.2 and 1.3, we may assume that $\mathbb{N}^* \setminus \{z\} = \bigcup_{\alpha \in \omega_2} Z_\alpha^*$ and that for each α , $F \restriction Z_\alpha^*$ is induced by an involution h_α .

Again let H be $\omega_2^{<\omega_1}$ -generic, $\lambda = \omega_2^V$, and $\mathfrak{F} \subset \mathbb{P}_1$ be \mathbb{P}_1 -generic over V . Assume that \dot{h} is a $\mathbb{P}(\mathcal{F})$ -name of a function from \mathbb{N} into \mathbb{N} . It suffices to show that no $f \in \mathfrak{F}$ forces that $\dot{h} \bmod \text{finite}$ extends each h_α ($\alpha \in \lambda$).

At the risk of being too incomplete, we leave to the reader the fact that Lemma 1.3 can be generalized to show that there is an $f \in \mathfrak{F}$ such that either $f \Vdash_{\mathbb{P}_1} \dot{h} \restriction Z_\alpha \notin V$, or there is a sequence $\{n_k : k \in \mathbb{N}\}$ as before. This is simply due to the fact that the \mathbb{P}_1 -name of the ultrafilter x_1 can be replaced by any \mathbb{P}_1 -name of an ultrafilter on \mathbb{N} . Similarly, Lemma 3.1 can be generalized in this setting to establish that there must be an $f \in \mathfrak{F}$ and a sequence of sets $\{m_k, S_k, T_k : k \in K \in [\mathbb{N}]^\omega\}$ with bijections $\psi : S_k \rightarrow T_k$ such that $S_k \subset (2^{m_k+1} \setminus 2^{m_k}) \subset [n_k, n_{k+1})$, $T_k \subset [n_k, n_{k+1})$, $\mathbb{N} \setminus \text{dom}(f) \subset \bigcup_k S_k$, and for each k and $i \in S_k$ and $\bar{f} < f$ \bar{f} forces a value on $\dot{h}(\psi(i))$ iff $i \in \text{dom}(\bar{f})$. The difference here is that we may have that $f \Vdash_{\mathbb{P}_1} \text{dom}(f) \subset Z_\alpha$, but there will be some values of \dot{h} not yet decided since $V[H]$ does not have a function extending all the h_α 's. Set $\Psi = \bigcup \psi$ which is a 1-to-1 function.

The contradiction now is that there will be some $f' < f$ such that $f' \Vdash_{\mathbb{P}_1} \Psi^*(x) \neq z$ (because we know that x is not a tie-point). Therefore we may assume that $\Psi(\text{dom}(f') \cap \text{dom}(\Psi))$ is a member of z and so that $\Psi(\text{dom}(\Psi) \setminus \text{dom}(f'))$ is not a member of z . By assumption, there is some $\bar{f} < f'$ and an $\alpha \in \lambda$ such that $\bar{f} \Vdash_{\mathbb{P}_1} \Psi(\text{dom}(\Psi) \setminus \text{dom}(f')) \subset Z_\alpha$. However this implies \bar{f} forces that $\dot{h}(\Psi(i)) = h_\alpha(\Psi(i))$ for almost all $i \in \bigcup_k S_k \setminus \text{dom}(\bar{f})$, contradicting that \bar{f} does not force a value on $\dot{h}(\Psi(i))$ for all $i \notin \bar{f}$.

4. QUESTIONS

Question 4.1. Assume PFA. If G is \mathbb{P}_2 -generic, and $\mathbb{N}^* = A \mathbin{\overset{x}{\boxtimes}} B$ is the generic tie-point introduced by \mathbb{P}_2 , is it true that A is not homeomorphic to \mathbb{N}^* ? Is it true that $\tau(x) = 2$? Is it true that each tie-point is a symmetric tie-point?

Remark 1. The tie-point x_3 introduced by \mathbb{P}_3 does not satisfy that $\tau(x_3) = 3$. This can be seen as follows. For each $f \in \mathbb{P}_3$, we can partition $\min(f)$ into $\{i \in \text{dom}(f) : i < f(i) < f^2(i)\}$ and $\{i \in \text{dom}(f) : i < f^2(i) < f(i)\}$.

It seems then that the tie-points x_ℓ introduced by \mathbb{P}_ℓ might be better characterized by the property that there is an autohomeomorphism F_ℓ of \mathbb{N}^* satisfying that $\text{fix}(F_\ell) = \{x_\ell\}$, and each $y \in \mathbb{N}^* \setminus \{x\}$ has an orbit of size ℓ .

Remark 2. A small modification to the poset \mathbb{P}_2 will result in a tie-point $\mathbb{N}^* = A \mathbin{\text{\tiny \times}}_x B$ such that A (hence the quotient space by the associated involution) is homeomorphic to \mathbb{N}^* . The modification is to build into the conditions a map from the pairs $\{i, f(i)\}$ into \mathbb{N} . A natural way to do this is the poset $f \in \mathbb{P}_2^+$ if f is a 2-to-1 function such that for each n , f maps $\text{dom}(f) \cap (2^{n+1} \setminus 2^n)$ into $2^n \setminus 2^{n-1}$, and again $\limsup_n |2^{n+1} \setminus (\text{dom}(f) \cup 2^n)| = \infty$. \mathbb{P}_2^+ is ordered by almost containment. The generic filter introduces an ω_2 -sequence $\{f_\alpha : \alpha \in \omega_2\}$ and two ultrafilters: $x \supset \{\mathbb{N} \setminus \text{dom}(f_\alpha) : \alpha \in \omega_2\}$ and $z \supset \{\mathbb{N} \setminus \text{range}(f_\alpha) : \alpha \in \omega_2\}$. For each α and $a_\alpha = \min(f_\alpha) = \{i \in \text{dom}(f_\alpha) : i = \min(f_\alpha^{-1}(f_\alpha(i)))\}$, we set $A = \{x\} \cup \bigcup_\alpha a_\alpha^*$ and $B = \{x\} \cup \bigcup_\alpha (\text{dom}(f_\alpha) \setminus a_\alpha)^*$, and we have that $\mathbb{N}^* = A \mathbin{\text{\tiny \times}}_x B$ is a symmetric tie-point. Finally, we have that $F : A \rightarrow \mathbb{N}^*$ defined by $F(x) = z$ and $F \upharpoonright A \setminus \{x\} = \bigcup_\alpha (f_\alpha)^*$ is a homeomorphism.

Question 4.2. Assume PFA. If L is a finite subset of \mathbb{N} and $\mathbb{P}_L = \Pi\{\mathbb{P}_\ell : \ell \in L\}$, is it true that in $V[G]$ that if x is tie-point, then $\tau(x) \in L$; and if $1 \notin L$, then every tie-point is a symmetric tie-point?

REFERENCES

- [1] Alan Dow, Petr Simon, and Jerry E. Vaughan, *Strong homology and the proper forcing axiom*, Proc. Amer. Math. Soc. **106** (1989), no. 3, 821–828. MR MR961403 (90a:55019)
- [2] Alan Dow and Geta Techanie, *Two-to-one continuous images of \mathbb{N}^** , Fund. Math. **186** (2005), no. 2, 177–192. MR MR2162384 (2006f:54003)
- [3] Ilijas Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702, xvi+177. MR MR1711328 (2001c:03076)
- [4] Ronnie Levy, *The weight of certain images of ω* , Topology Appl. **153** (2006), no. 13, 2272–2277. MR MR2238730 (2007e:54034)
- [5] S. Shelah and J. Steprāns. Non-trivial homeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ without the Continuum Hypothesis. *Fund. Math.*, 132:135–141, 1989.
- [6] S. Shelah and J. Steprāns. Somewhere trivial autohomeomorphisms. *J. London Math. Soc. (2)*, 49:569–580, 1994.

- [7] Saharon Shelah and Juris Steprāns, *Martin's axiom is consistent with the existence of nowhere trivial automorphisms*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 2097–2106 (electronic). MR 1896046 (2003k:03063)
- [8] Juris Steprāns, *The autohomeomorphism group of the Čech-Stone compactification of the integers*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4223–4240 (electronic). MR 1990584 (2004e:03087)
- [9] B. Velickovic. Definable automorphisms of $\mathcal{P}(\omega)/\text{fin}$. *Proc. Amer. Math. Soc.*, 96:130–135, 1986.
- [10] Boban Veličković. OCA and automorphisms of $\mathcal{P}(\omega)/\text{fin}$. *Topology Appl.*, 49(1):1–13, 1993.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER, PISCATAWAY, NEW JERSEY, U.S.A. 08854-8019

Current address: Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel

E-mail address: `shelah@math.rutgers.edu`